ELEN E3401: Electromagnetics

Spring 2025

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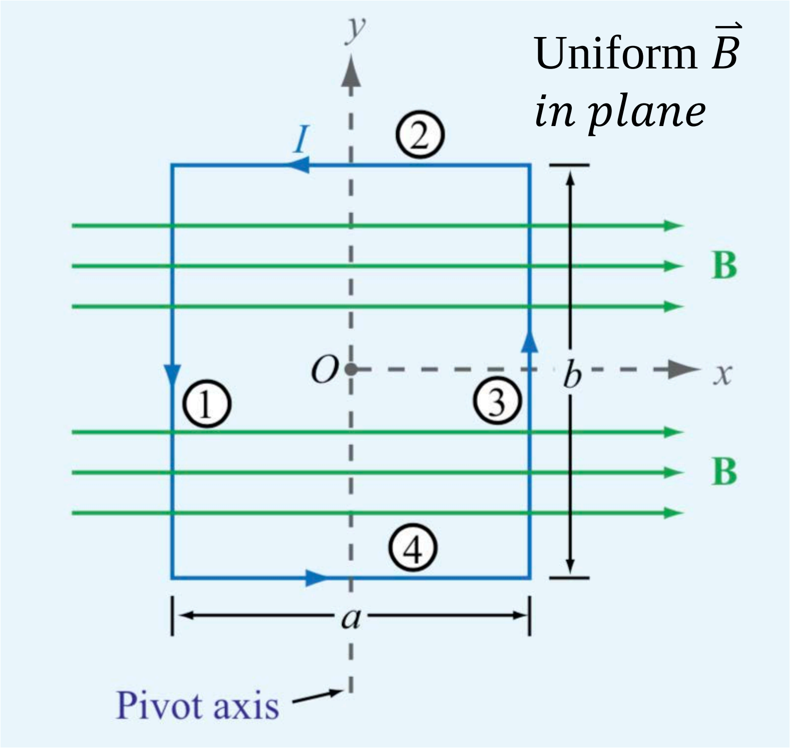
Lecture #15

COLUMBIA | ENGINEERING
The Fu Foundation School of Engineering and Applied Science

Magnetic torque on current carrying loop

Consider loop in x-y plane, $\vec{B} = \hat{x}B_0$



Force on each of the 4 wire segments:

$$\vec{F}_1 = I(-\hat{y}b) \times \hat{x}(B_0) = \hat{z}IbB_0$$

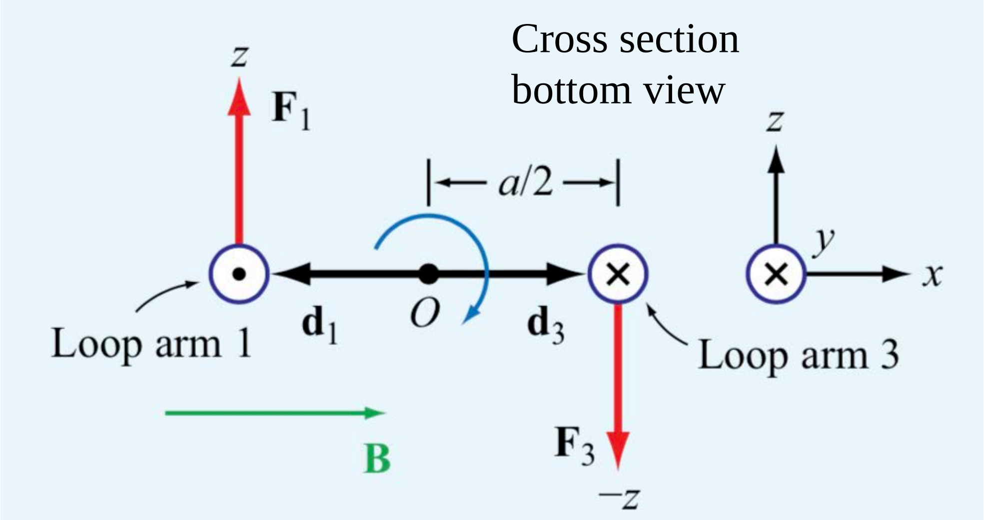
$$\vec{F}_3 = I(\hat{y}b) \times \hat{x}(B_0) = -\hat{z}IbB_0$$

$$\vec{F}_2 = 0 \qquad \vec{F}_4 = 0$$

$\longrightarrow$ $\vec{B}$ is parallel to I

Magnetic torque on current carrying loop

Consider loop in x-y plane, $\vec{B} = \hat{x}B_0$



Total force is 0 but -
$\vec{F}_1$ and $\vec{F}_3$ produce a torque
loop will rotate clockwise

Moment arm → $a/2$ $\quad \vec{d}_1 = -\hat{x}\dfrac{a}{2} \qquad \vec{d}_3 = +\hat{x}\dfrac{a}{2}$

$$\vec{T} = \vec{d}_1 \times \vec{F}_1 + \vec{d}_3 \times \vec{F}_3$$

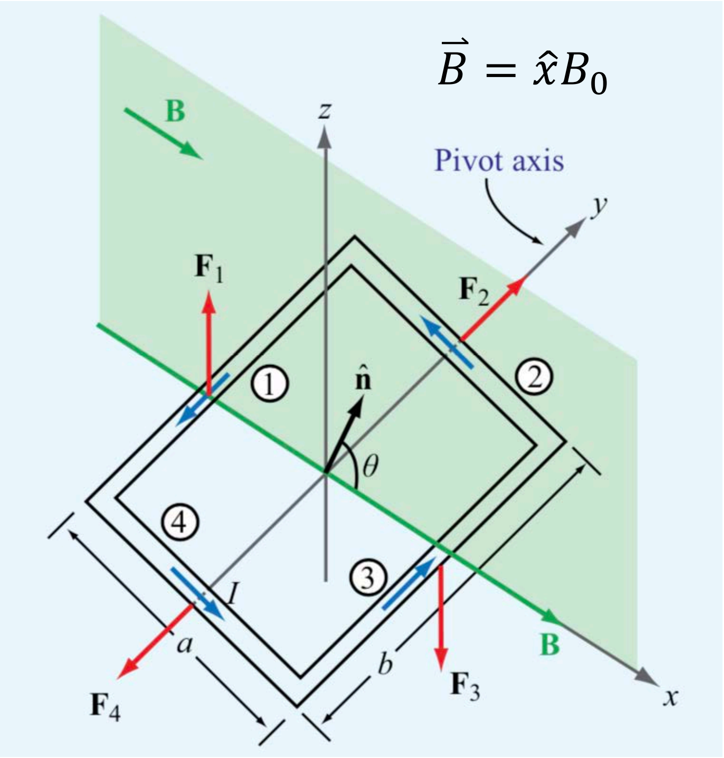
$$\vec{T} = (-\hat{x}\frac{a}{2}) \times (\hat{z}IbB_0) + (\hat{x}\frac{a}{2}) \times (-\hat{z}IbB_0)$$

$$\vec{T} = \hat{y}IabB_0 = \hat{y}IAB_0 \qquad A = ab \ \rightarrow loop\ cross\ sectional\ area$$

Rectangular Loop

Now the loop rotates

but after ¼ turn: Torque → 0



$\vec{F}_2$ and $\vec{F}_4$ still along rotation axis

Only $\vec{F}_1$ and $\vec{F}_3$ contribute to torque

$\hat{n}$ = surface normal to loop

RHR: fingers around direction of current and thumb along direction of $\hat{n}$

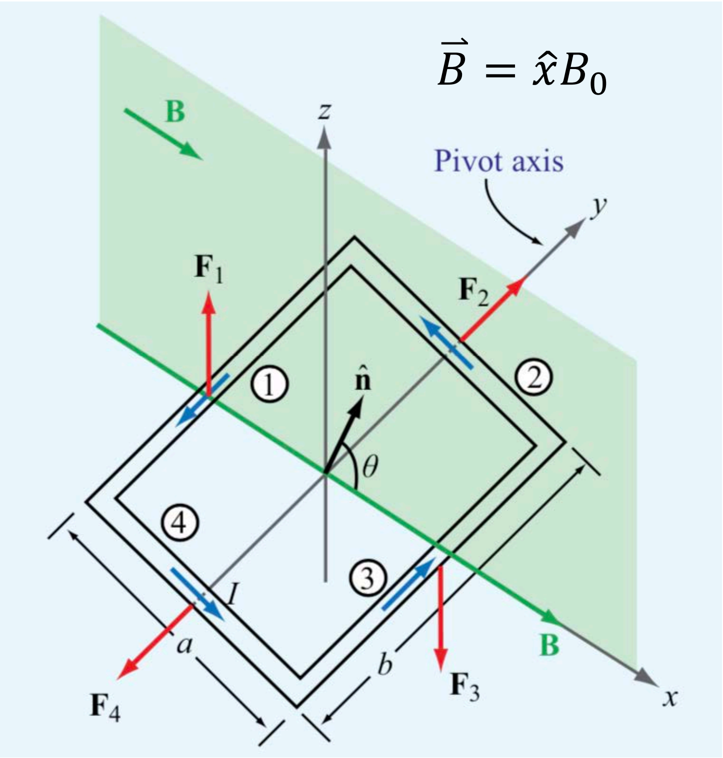
<u>moment arm</u> = $(\frac{a}{2})\sin\theta$

Net torque: $T = IAB_0\sin\theta$ $\qquad \vec{B} = \hat{x}B_0$

$\theta = 90^\circ \rightarrow \vec{B}$ is parallel to loop; Torque max

$\theta = 0^\circ \rightarrow \vec{B}$ is perpendicular to loop; Torque = 0

Rectangular Loop



moment arm = $(\frac{a}{2})\sin\theta$ $\vec{B} = \hat{x}B_0$

Net torque: $T = IAB_0\sin\theta$

$\theta = 90^\circ \rightarrow \vec{B}$ is parallel to loop; Torque max

$\theta = 0^\circ \rightarrow \vec{B}$ is perpendicular to loop; Torque = 0

If we add N turns of wire:

$$T = \underbrace{NIA}B_0\sin\theta$$

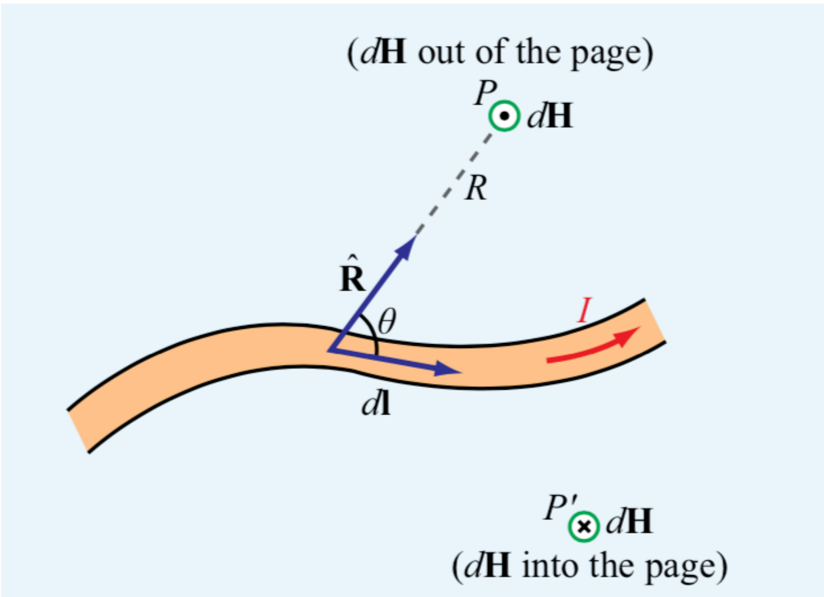
"magnetic moment" magnitude

$$\vec{m} = \hat{n}NIA = \hat{n}\text{m}$$

$\vec{T} = \vec{m} \times \vec{B}$ [N ·m] And applies to any shape, any direction $\vec{B}$

Biot-Savart Law

Jean Biot + Felix Savart → relate magnetic field $\vec{H}$ to its generating current, $\vec{I}$



Magnetic field, $d\vec{H}$ generated by current element $Id\vec{l}$

$d\vec{l}$: along direction of current flow

$\hat{R}$: points from current element to observation point

$d\vec{H}$ varies with: $\frac{1}{R^2}$ (similar to $\vec{E}$)

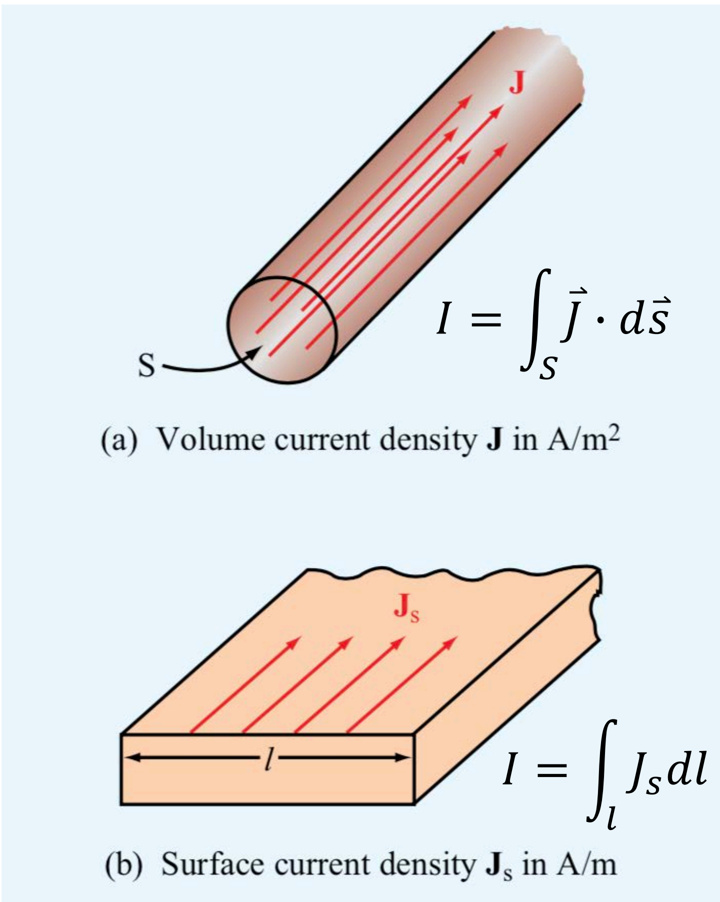
Note:
$\vec{E}$ points along $\vec{R}$
$\vec{H}$ is orthogonal to $d\vec{l}$ and $\hat{R}$

$$d\vec{H} = \frac{I}{4\pi} \frac{d\vec{l} \times \hat{R}}{R^2} \quad [\mathrm{A/m}] \qquad (\vec{R} = \hat{R}R)$$

$d\vec{H}$ generated by steady current, I flowing through $d\vec{l}$

Magnetic field due to current and current distributions

To determine total magnetic field, $\vec{H}$ → sum up contributions of current elements:



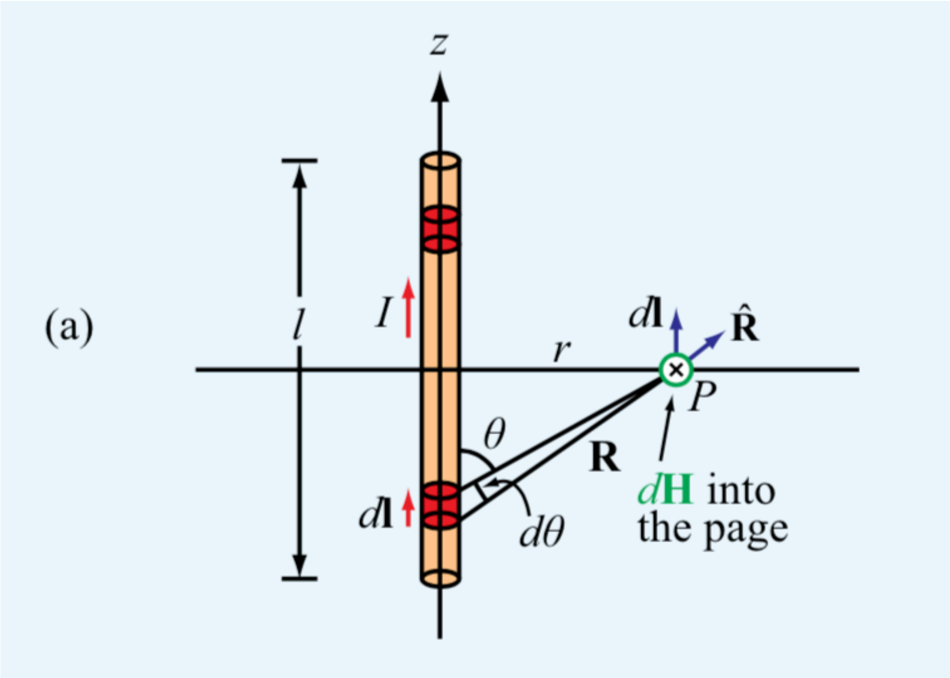
$$\boxed{\vec{H} = \frac{I}{4\pi} \int_l \frac{d\vec{l} \times \hat{R}}{R^2}}$$

$I d\vec{l} \rightarrow \vec{J}_s ds \rightarrow \vec{J} d\mathcal{V}$ (volume)

$$\vec{H} = \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\vec{J} \times \hat{R}}{R^2} d\mathcal{V}$$ Volume current

$$\vec{H} = \frac{1}{4\pi} \int_S \frac{\vec{J}_s \times \hat{R}}{R^2} ds$$ Surface current

Magnetic field of linear conductor



Conductor length, l carrying current I along $+\hat{z}$

Determine $\vec{B}$ at point, P, located distance r in the x-y plane

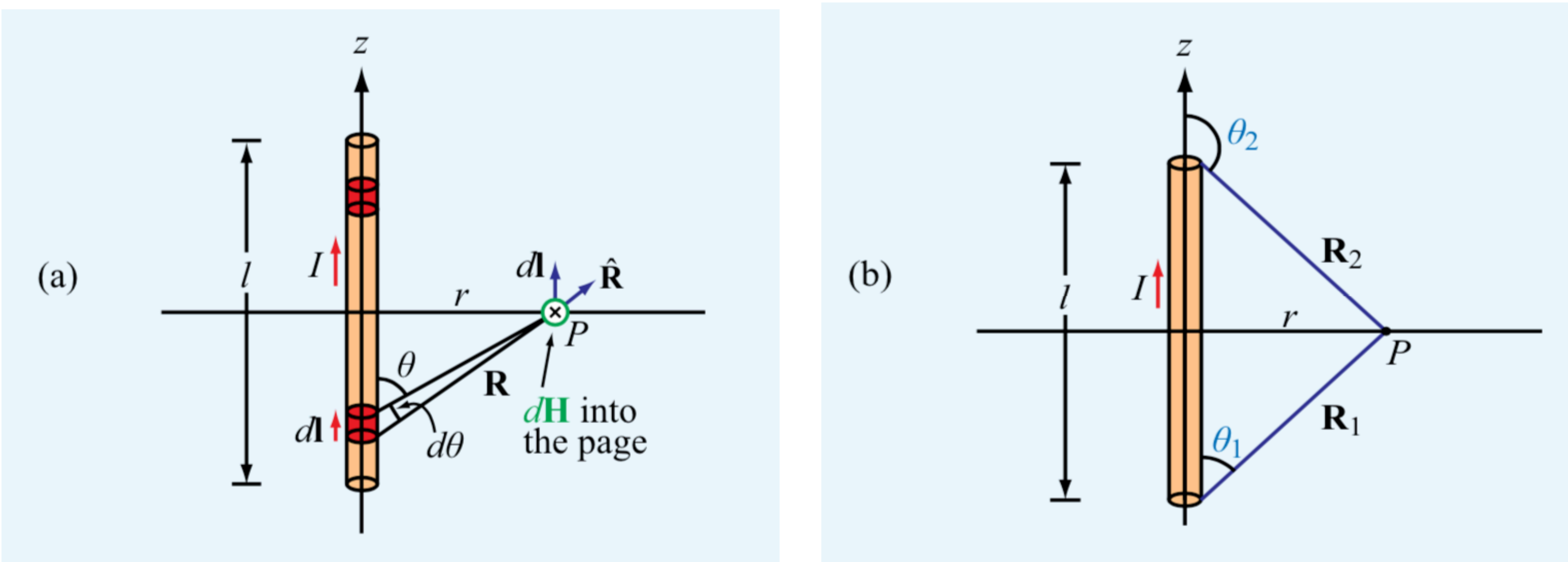
$$\vec{H} = \frac{I}{4\pi}\int_l \frac{d\vec{l} \times \hat{R}}{R^2}$$

Differential length vector: $d\vec{l} = \hat{z}dz$

$$d\vec{l} \times \hat{R} = dz(\hat{z} \times \hat{R}) = \hat{\varphi}\underbrace{sin\theta}_{}dz$$

$\theta =$ angle between $d\vec{l}$ and $\hat{R}$

Magnetic field of linear conductor



Limiting angles θ_1, θ_2 each measured between vector $I d\vec{l}$ and vector connecting end of conductor associated with that angle to point P

$$\vec{H} = \frac{I}{4\pi}\int_{z=-l/2}^{z=l/2} \frac{d\vec{l} \times \hat{R}}{R^2} = \hat{\varphi}\,\frac{I}{4\pi}\int_{-l/2}^{l/2} \frac{sin\theta}{R^2}\,dz$$

Both R and θ depend on z

Magnetic field of linear conductor

$$\vec{H} = \hat{\varphi}\frac{I}{4\pi}\int_{-l/2}^{l/2}\frac{sin\theta}{R^2}dz$$

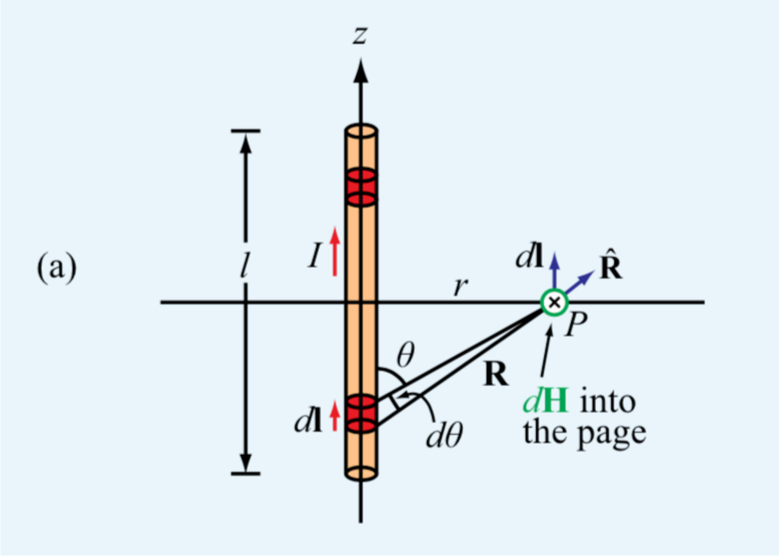
both R and θ depend on z

$$csc\theta = \frac{1}{sin\theta}$$

We change integration variable from z → to → θ

$$tan\theta = \frac{-r}{z}$$

$$R = rcsc\theta \qquad z = -rcot\theta \qquad dz = rcsc^2\theta d\theta \qquad sin\theta = \frac{r}{R}$$



$$\vec{H} = \hat{\varphi}\frac{I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{sin\theta rcsc^2\theta d\theta}{r^2csc^2\theta}$$

$$\vec{H} = \hat{\varphi}\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}sin\theta d\theta$$

$$= \hat{\varphi}\frac{I}{4\pi r}(cos\theta_1 - cos\theta_2)$$

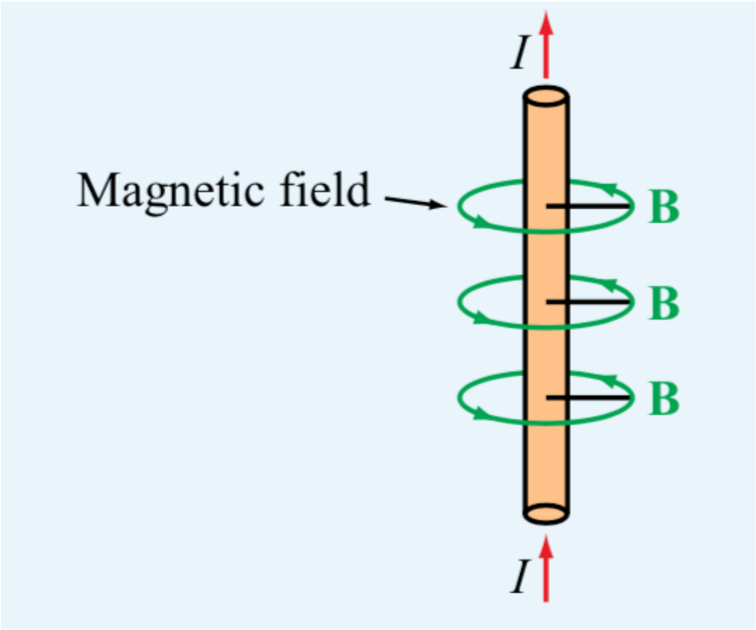
$$cos\theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}} \qquad cos\theta_2 = -cos\theta_1 = \frac{-l/2}{\sqrt{r^2 + (l/2)^2}}$$

Magnetic field of linear conductor

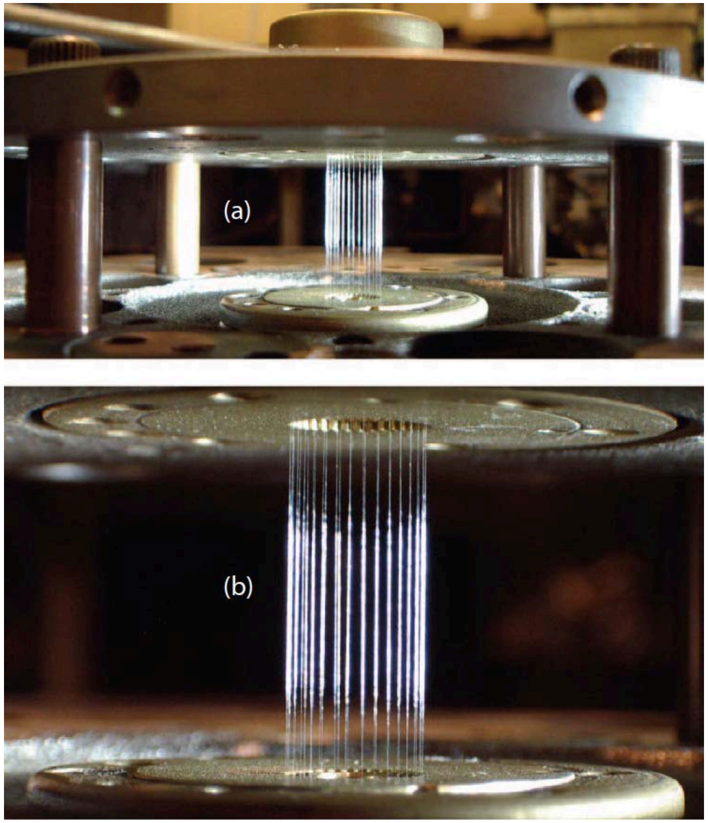
$$\vec{B} = \mu_0 \vec{H} = \hat{\varphi} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}}$$

For infinite long wire, $l \gg r \rightarrow \quad \vec{B} = \hat{\varphi} \frac{\mu_0 I}{2\pi r}$

$\vec{B}$ forms concentric circles around wire, with magnitude ~ $\frac{1}{r}$



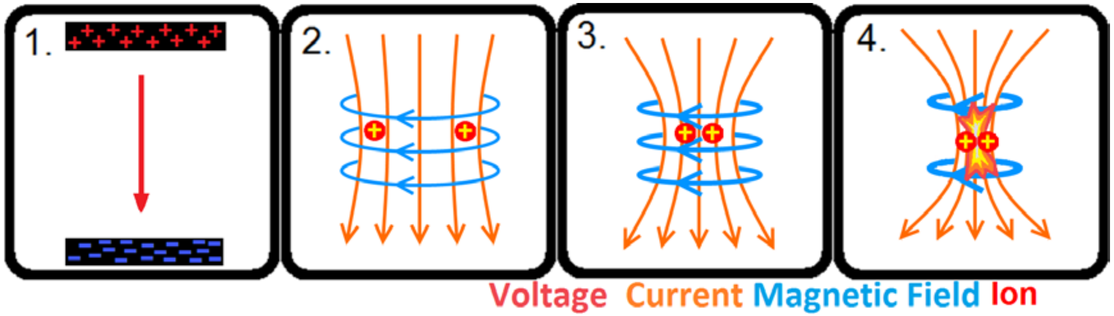
Example: Lorentz Force for High Energy Density Plasma Research



- 1 Mega Amp pulse over ~300 ns
- Wire array instantly ablates into plasma
- Superposition of magnetic field lines
  - Strong magnetic field on the outside
  - Magnetic “vacuum” on the inside

$$\vec{F} = q\vec{u} \times \vec{B}$$

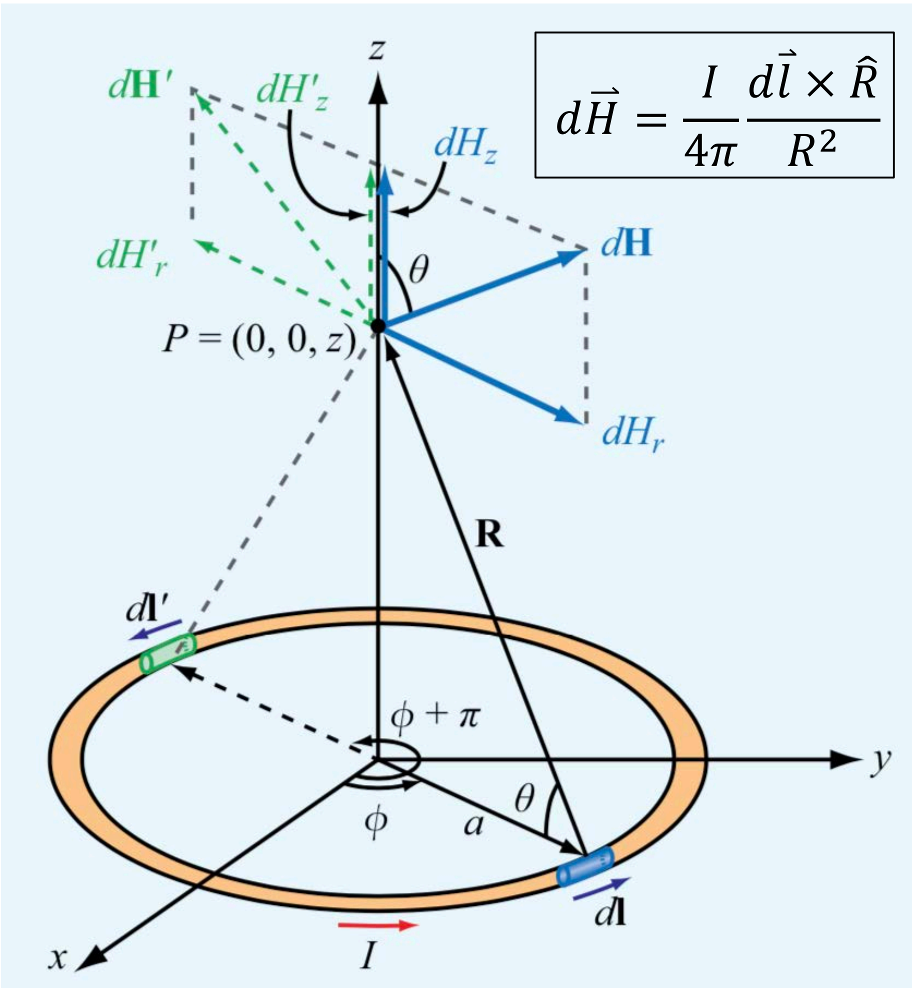
- Violent plasma cylinder implosion!
  - Variations used for studying wide range of topics, from astrophysics to nuclear fusion



For more details visit:
https://www.lps.cornell.edu/project/cobra/
https://en.wikipedia.org/wiki/Pinch_(plasma_physics)

Magnetic field of circular loop: Biot-Savart Law

Consider circular loop, radius = a carrying I → Find $\vec{H}$ at a point on axis.



Loop on x-y plane

$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l} \times \hat{R}}{R^2}$$

* $d\vec{l}$ is ⊥ to $\vec{R}$

* All elements around loop are <u>same</u> distance to P: $R = \sqrt{a^2 + z^2}$

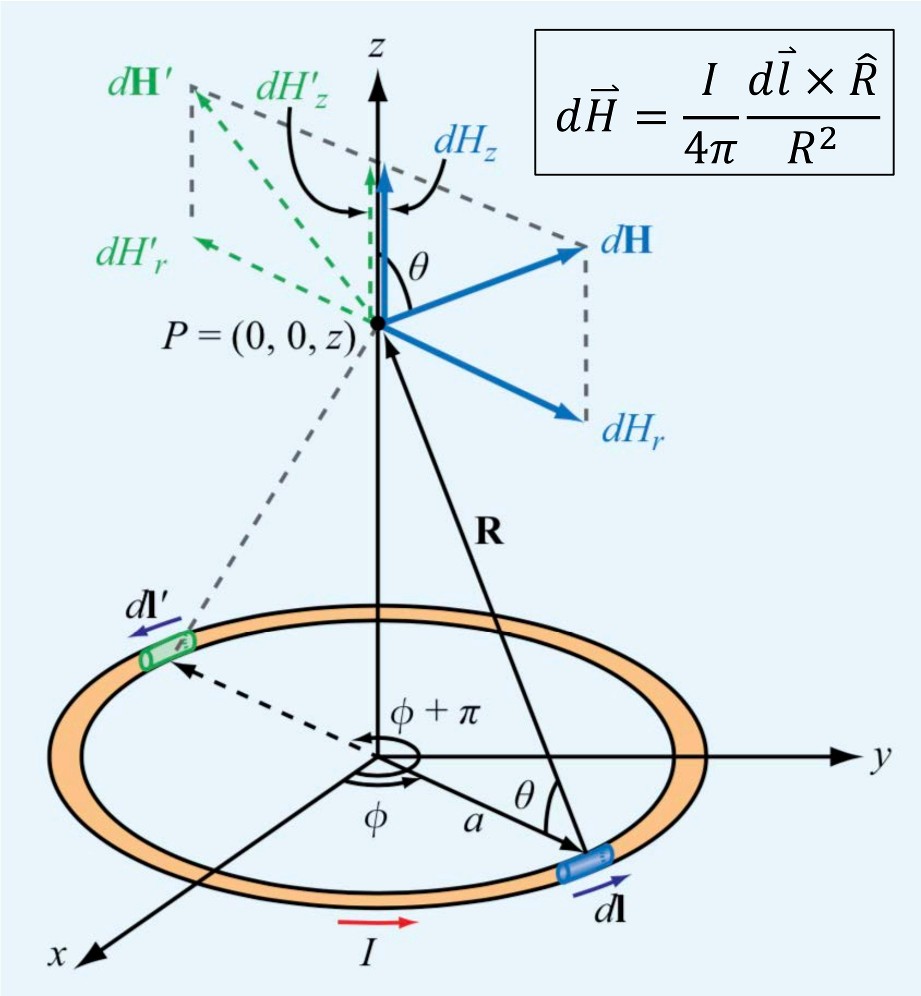
$$dH = \frac{I}{4\pi R^2}\left|d\vec{l} \times \hat{R}\right| = \frac{Idl}{4\pi(a^2 + z^2)}$$

$d\vec{H}$ is ⊥ to plane containing $\vec{R}$ and $d\vec{l}$

$d\vec{H}$: in the r-z plane
→ will have dH_r and dH_z

Note: the z-component of $d\vec{H}$ will <u>add</u> from $d\vec{l}$, $d\vec{l}'$
but r-components <u>cancel</u>

Magnetic field of circular loop: Biot-Savart Law



$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l} \times \hat{R}}{R^2}$$

$$\underbrace{dH'_r = -dH_r}_{\text{cancel}} \qquad \underbrace{dH'_z = dH_z}_{\text{add}}$$

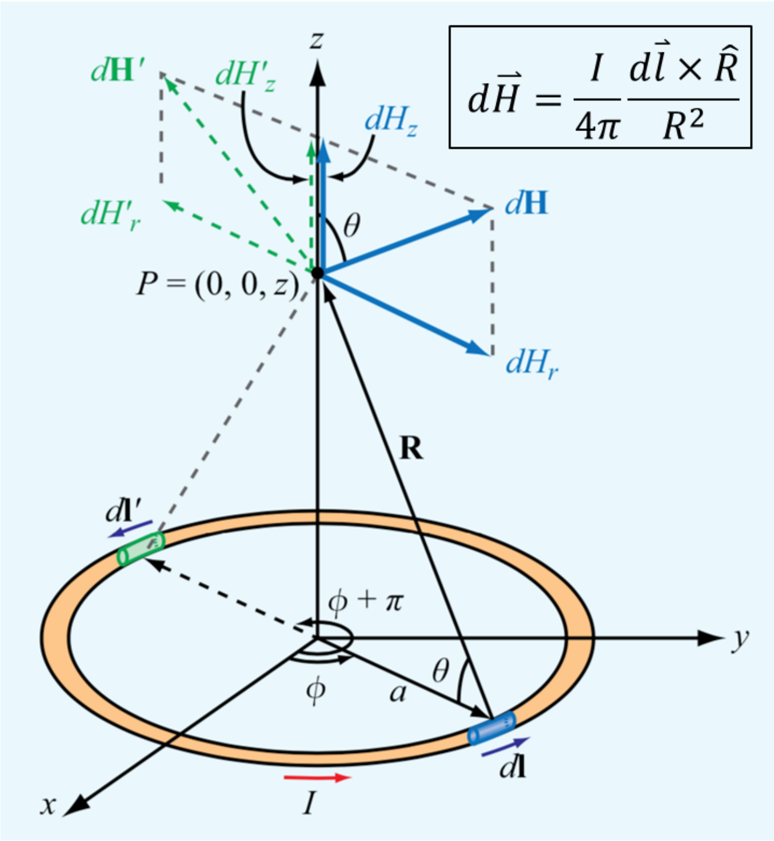
Only dH_z components remain:

$$d\vec{H} = \hat{z}dH_z = \hat{z}(\overbrace{dH\cos\theta}^{dH_z})$$

$$= \hat{z}\frac{I\cos\theta}{4\pi(a^2+z^2)}dl$$

$$d\vec{H} = \hat{z}\frac{I\cos\theta}{4\pi(a^2+z^2)}dl$$

Magnetic field of circular loop: Biot-Savart Law



At point P(0,0,z) all quantities are constant, except for *dl* → so we integrate over *dl*:

$$\vec{H} = \hat{z}\frac{I\cos\theta}{4\pi(a^2+z^2)}\oint_{over\ circle} dl$$

$$\vec{H} = \hat{z}\frac{I\cos\theta}{4\pi(a^2+z^2)}(2\pi a)$$

using $\cos\theta = \frac{a}{\sqrt{a^2+z^2}}$

$$\boxed{\vec{H} = \hat{z}\frac{Ia^2}{2(a^2+z^2)^{3/2}}}$$ [A/m]

At z = 0 (center of loop):

$$\vec{H} = \hat{z}\frac{I}{2a}$$

For $z^2 \gg a^2$ (far from loop):

$$\vec{H} = \hat{z}\frac{Ia^2}{2|z|^3}$$

Electrostatics and Magnetostatics

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations • Differential form • Integral form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$ $\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$ $\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Gauss's Law for Magnetism

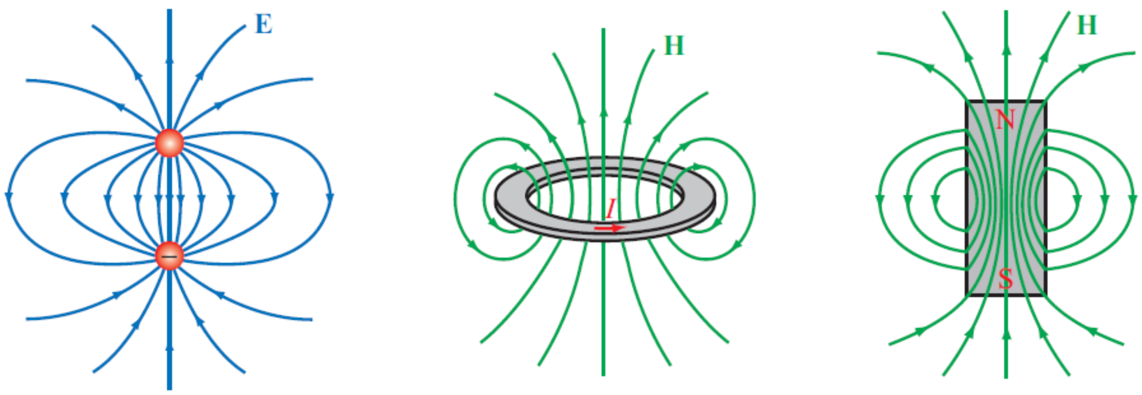
Gauss - Electric $\quad \vec{\nabla} \cdot \vec{D} = \rho_V \longleftrightarrow \oint_S \vec{D} \cdot d\vec{s} = Q$

Similarly: $\quad \vec{\nabla} \cdot \vec{B} = 0 \longleftrightarrow \oint_S \vec{B} \cdot d\vec{s} = 0$

Divergence theorem

No magnetic monopole

→ magnetic field lines always form continuous closed loops



(a) Electric dipole (b) Magnetic dipole (c) Bar magnet

Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a *magnetic dipole*

Ampère's Law

Electrostatic field conservative:

$$\vec{\nabla} \times \vec{E} = 0 \quad \longleftrightarrow \quad \oint_C \vec{E} \cdot d\vec{l} = 0$$

Stoke's theorem

Surface *S*, contour *C*

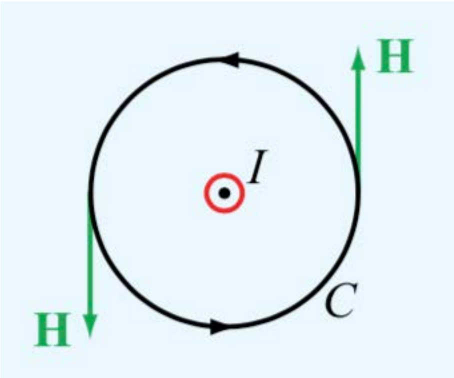
Ampère's Law:

$$\vec{\nabla} \times \vec{H} = \vec{J} \quad \longleftrightarrow \quad \oint_C \vec{H} \cdot d\vec{l} = I$$

Total current through surface, *S*

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{s} = \int_S \vec{J} \cdot d\vec{s} = I$$

Contour follows RHR: *I* along thumb, the contour follows RHR



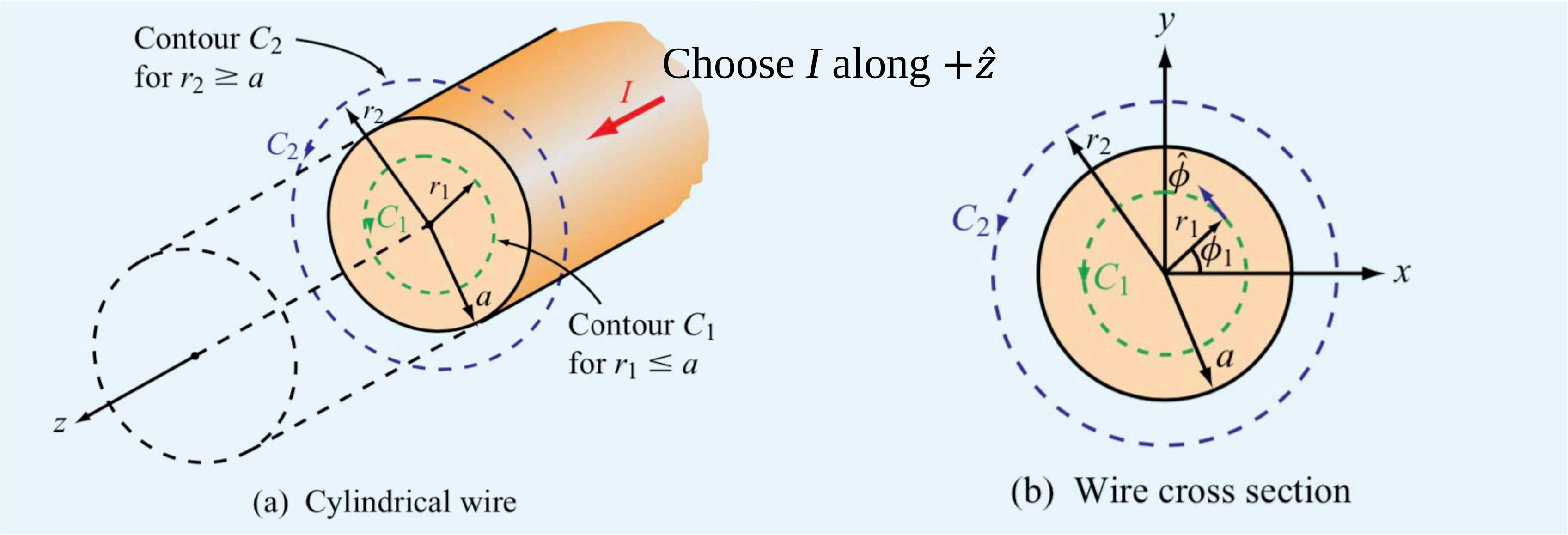
We use symmetry in applying contours, similar to Gaussian surfaces

Magnetic field of long wires

Consider very long straight wire of radius a, carrying a steady current I uniformly distributed over cross section.

Find $\vec{H}$ a distance r from wire axis for:
a) $r \leq a$ (inside wire)
b) $r \geq a$ (outside wire)

Choose I along $+\hat{z}$

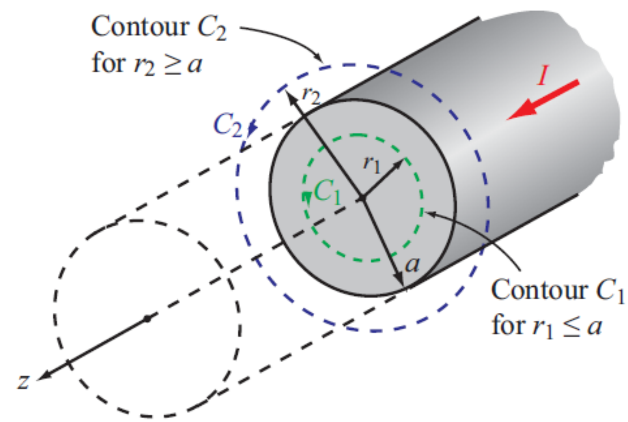


Magnetic field of long wires

1) I along $+\hat{z}$. $\vec{H}_1 = \vec{H}$ at $r = r_1 \leq a$

Choose Amperian contour, C_1 as circle of radius, $r = r_1$

$$\oint_{C_1} \vec{H}_1 \cdot d\vec{l}_1 = I_1 \longleftarrow$$ Fraction of total current flowing within C_1

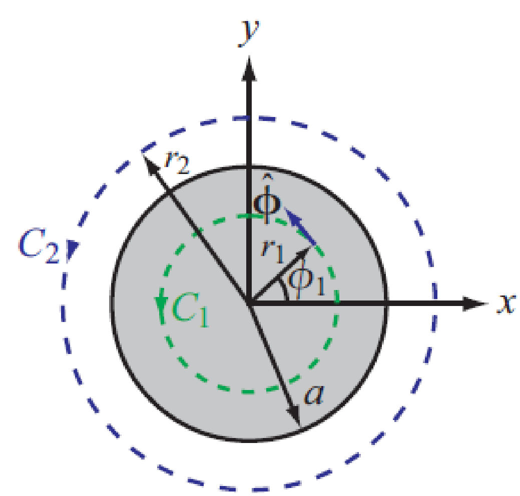


$\vec{H}_1$ by symmetry is constant and parallel to contour

Since I is along $+\hat{z}$, $\vec{H}_1$ must be along $+\hat{\varphi}$

$$\vec{H}_1 = \hat{\varphi} H_1 \qquad d\vec{l}_1 = \hat{\varphi} r_1 d\varphi$$

$$\oint_{C_1} \vec{H}_1 \cdot d\vec{l}_1 = \int_0^{2\pi} H_1(\hat{\varphi} \cdot \hat{\varphi}) r_1 d\varphi = 2\pi r_1 H_1$$



$$I_1 = \left(\frac{\pi r_1{}^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I$$

$$\boxed{\vec{H}_1 = \hat{\varphi} H_1 = \hat{\varphi} \frac{r_1}{2\pi a^2} I \qquad r_1 \leq a}$$

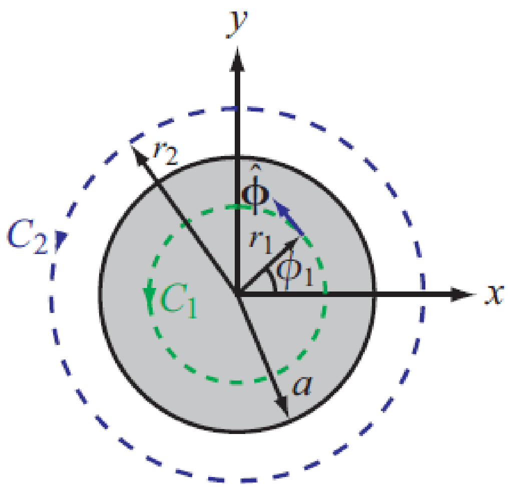
Example: Magnetic field of long wires

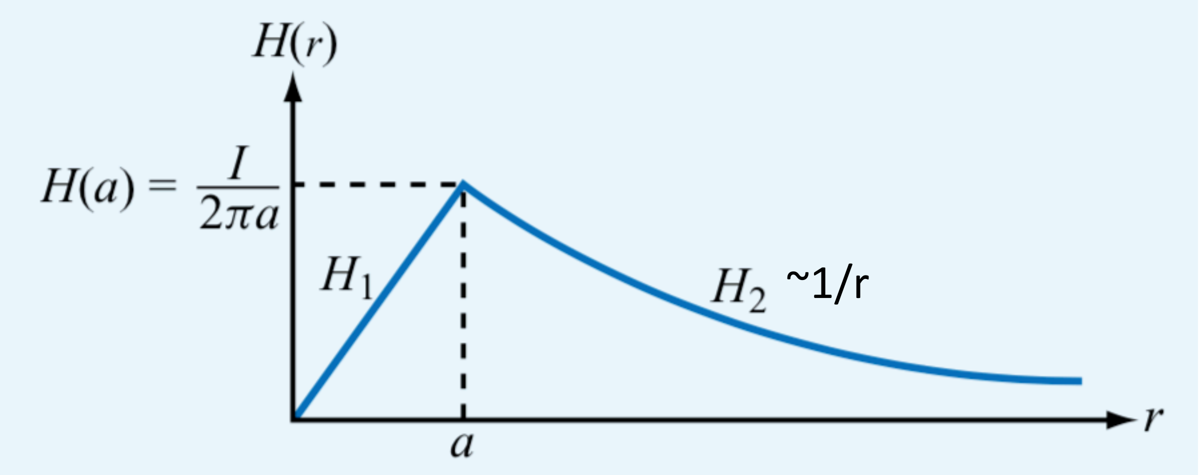
2) $r = r_2 \geq a$ Choose C_2, outside wire.

$$\vec{H}_2 = \hat{\varphi} H_2 \qquad d\vec{l}_2 = \hat{\varphi} r_2 d\varphi$$

$$\oint_{C_2} \vec{H}_2 \cdot d\vec{l}_2 = 2\pi r_2 H_2 = I$$

$$\boxed{\vec{H}_2 = \hat{\varphi} H_2 = \hat{\varphi} \frac{I}{2\pi r_2} \qquad r_2 \geq a}$$

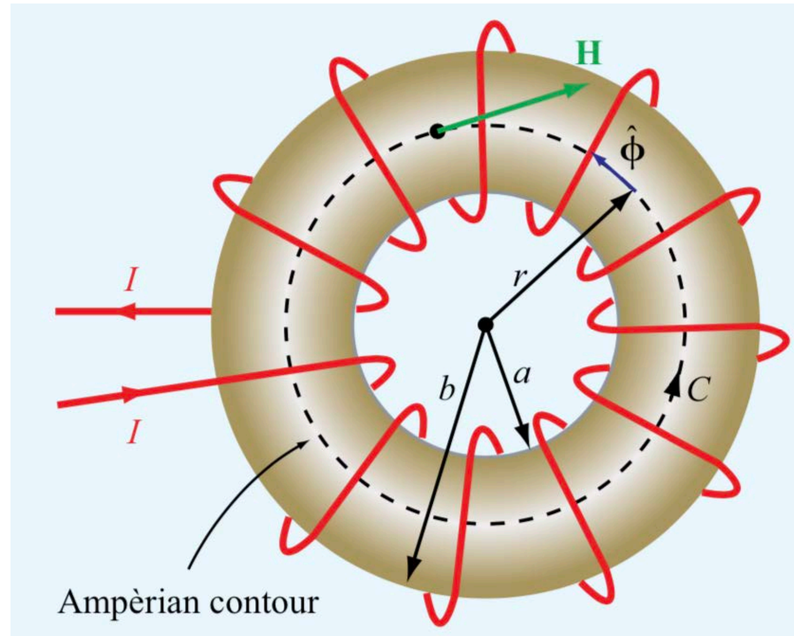




Magnetic Field of Toroid

Applying Ampere's law over contour C

$$\oint_C \vec{H} \cdot d\vec{l} = I$$



Ampere's law: the line integral of $\vec{H}$ around a closed contour C is equal to the current traversing the surface bounded by the contour.

What's the magnetic field outside the toroid?

$$\oint_C \vec{H} \cdot d\vec{l} = \int_0^{2\pi} (-\hat{\varphi}H) \cdot \hat{\varphi} r d\varphi = -2\pi r H = -NI$$

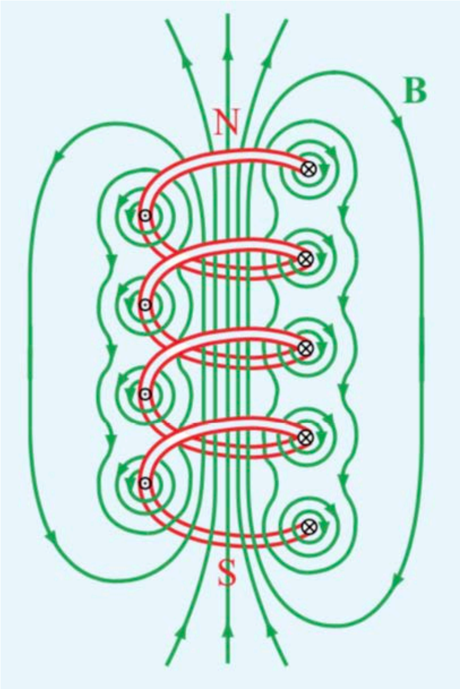
Hence, $H = -\frac{NI}{2\pi r}$ and $\vec{H} = -\hat{\varphi}H = -\hat{\varphi}\frac{NI}{2\pi r}$ for $a < r < b$

Inductance

Inductance → analog of capacitance

Capacitor can store energy in $\vec{E}$ field between conducting surfaces
Inductor stores energy in $\vec{H}$ field near current carrying conductors

<u>Solenoid</u>



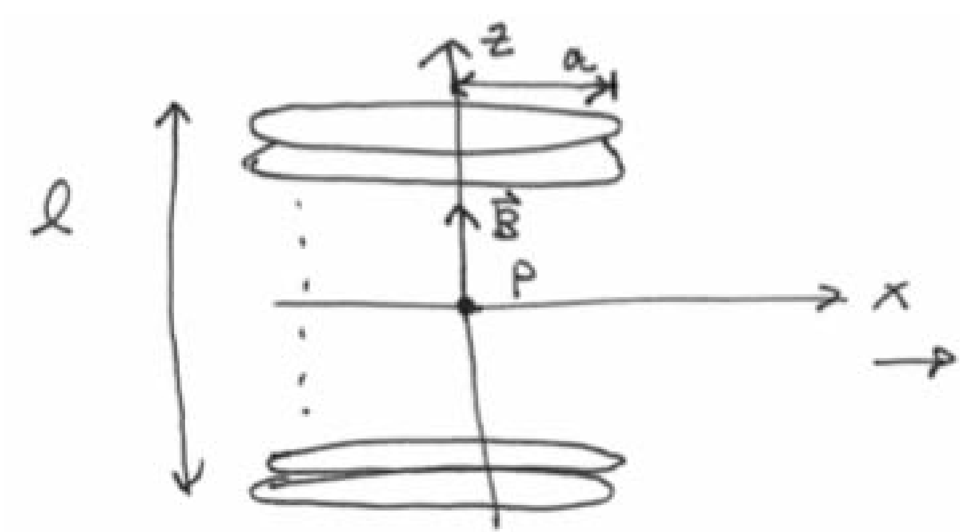
Multiple turns of wire – core can be air or magnetic material, μ

If turns are closely spaced
→ magnetic field can be uniform
→ looks more like permanent magnet

Magnetic Flux in Solenoid

Tightly wound solenoid of length = l
Number of turns / unit length: $n = \frac{N}{l}$

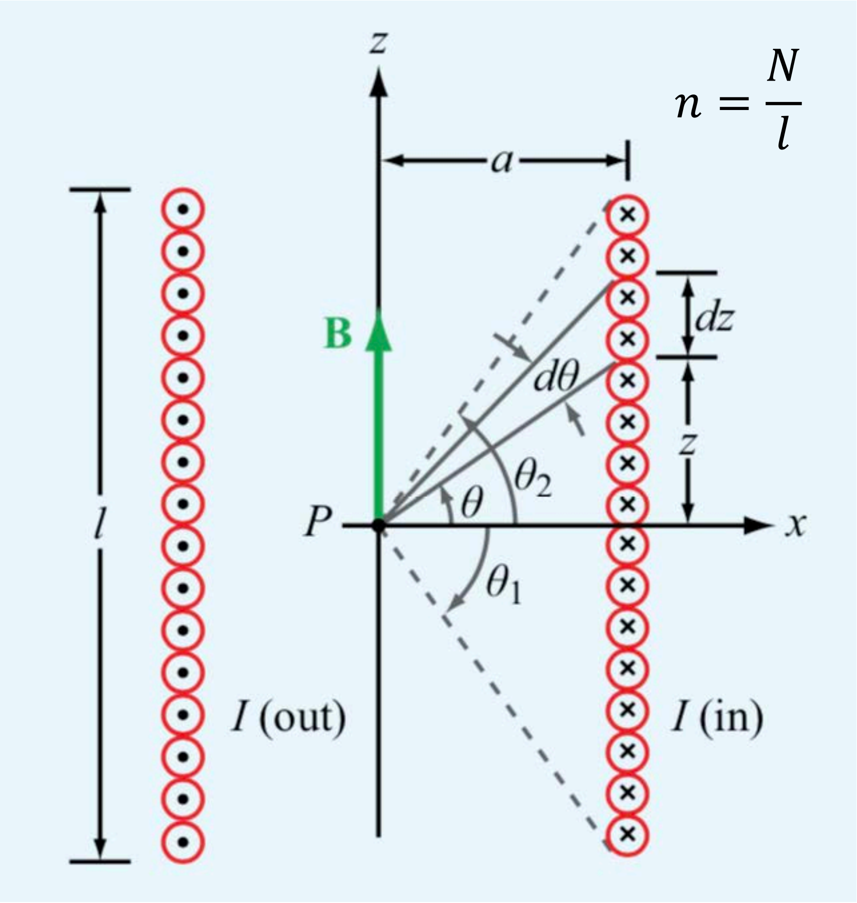
because tight wound, treat helical shape as rings



Radius = a
Length = l

Consider $\vec{B}$ at point P on axis

Solenoid



We previously obtained magnetic field, $\vec{H}$ along axis for 1 loop of current:

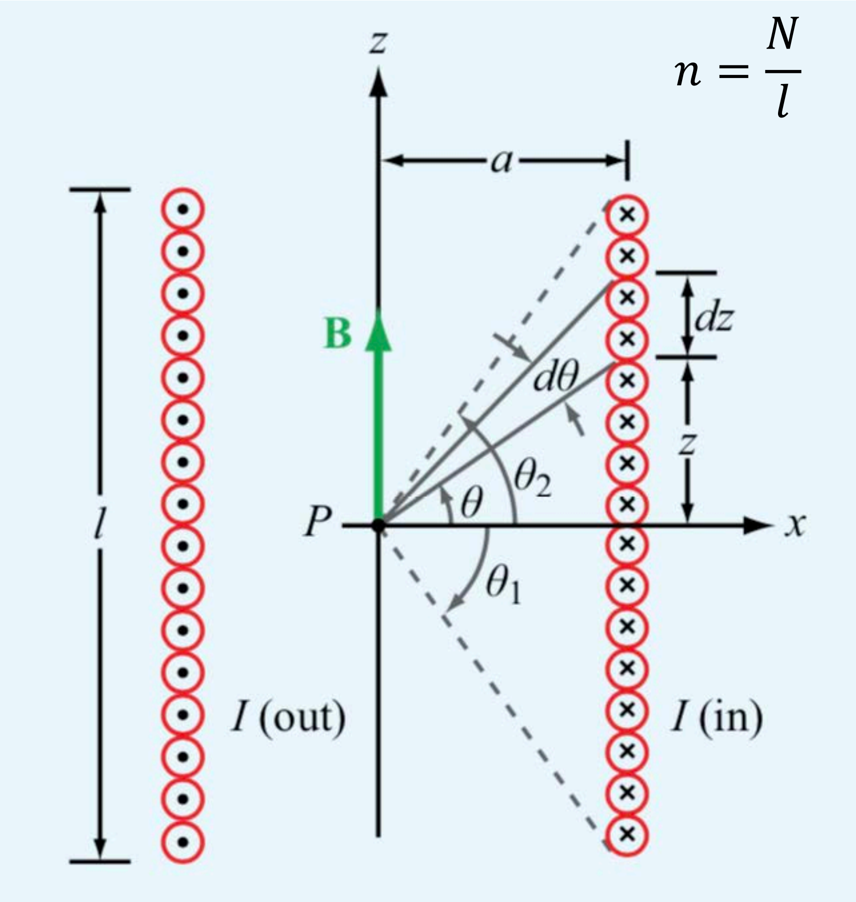
$$\vec{H} = \hat{z}\frac{I'a^2}{2(a^2+z^2)^{3/2}}$$

a = loop radius
I' = current in loop

$$n = \frac{N}{l}$$

For solenoid:
dz of solenoid = equivalent to loop composed of (*ndz*) turns carrying current
I' = I(ndz)

Solenoid



Then, at P:

$I' = Indz$

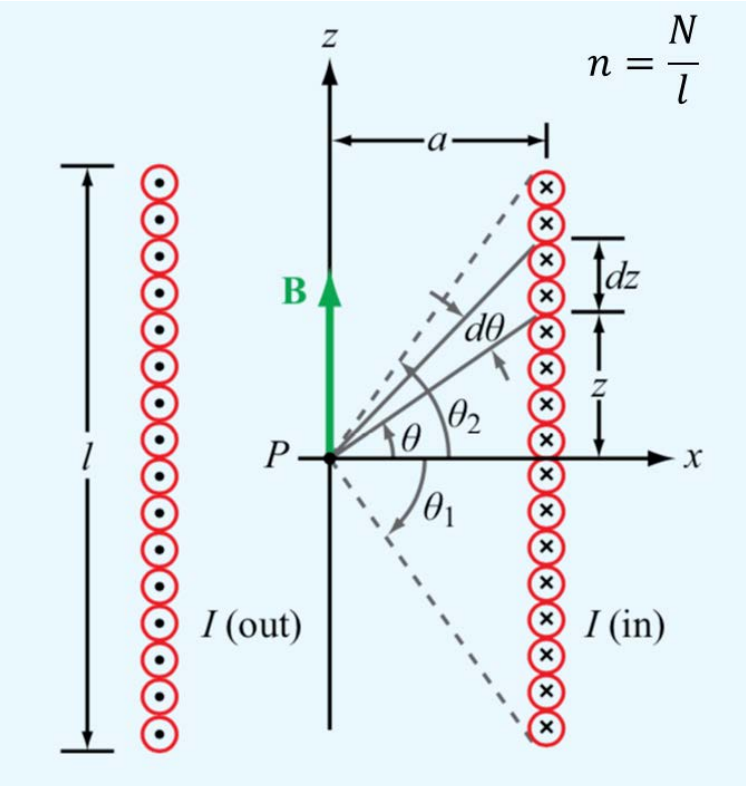
$$d\vec{B} = \mu d\vec{H} = \hat{z}\frac{\overbrace{\mu n I}\, a^2}{2(a^2+z^2)^{3/2}}dz$$

Total $\vec{B}$: integrate over length of solenoid to compute $\vec{B}$ as seen at P

Angle θ: angle as seen from P of solenoid

$$z = (a)tan\theta \leftarrow tan\theta = \frac{z}{a}$$

Solenoid



$$d\vec{B} = \mu d\vec{H} = \hat{z}\frac{\mu n I a^2}{2(a^2+z^2)^{3/2}}dz$$

$$z = (a)tan\theta$$

$$a^2 + z^2 = \underbrace{a^2 + a^2tan^2(\theta)}_{a^2(1+tan^2(\theta))} = a^2sec^2\theta$$

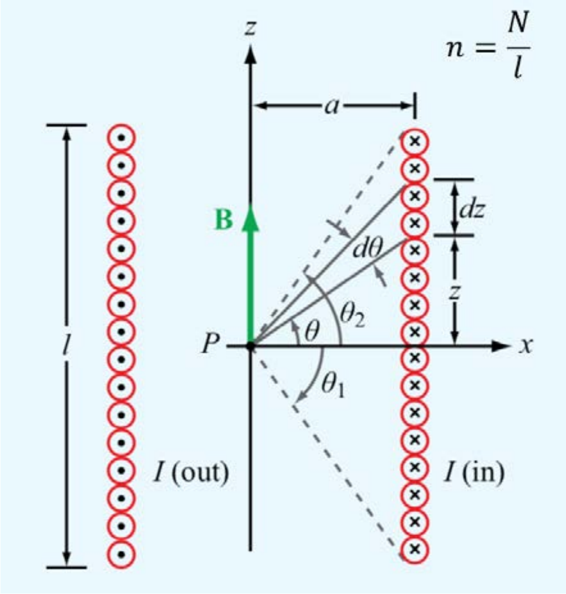
$$dz = asec^2\theta d\theta$$

$$d\vec{B} = \mu d\vec{H} = \hat{z}\frac{\mu n I a^2}{2(a^2+z^2)^{3/2}}dz$$

$$\vec{B} = \hat{z}\frac{\mu n I a^2}{2}\int_{\theta_1}^{\theta_2}\frac{asec^2\theta d\theta}{a^3sec^3\theta} \quad \begin{matrix} = dz \\ = (a^2+z^2)^{3/2} \end{matrix}$$

$$\vec{B} = \hat{z}\frac{\mu n I}{2}(sin\theta_2 - sin\theta_1)$$

Solenoid



$$\vec{B} = \hat{z}\frac{\mu n I}{2}(sin\theta_2 - sin\theta_1)$$

If solenoid length $l >> a$, then $\theta_1 = -\theta_2 \approx \pm 90°$ $\theta_1 \approx -90°$ $\theta_2 \approx 90°$

$$\vec{B} \approx \hat{z}\mu n I = \frac{\hat{z}\mu N I}{l}$$

Long solenoid $l\,/a >> 1$

Holds for $\vec{B}$ everywhere inside solenoid (except for ends)